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Regarding modules over infinite group rings

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Abstract Let R be a commutative ring and Γ represent an infinite discrete group. The computation of the algebraic K-theory of the group ring R[Γ] is a significant task in the fields of geometric topology and number theory. If the group ring is Noetherian, there exists a companion G-theory of R[Γ] that is often more computationally tractable. Nevertheless, it is quite uncommon for the group ring to be Noetherian when dealing with an infinite group. This work presents a definition of G-theory that is applicable to any discrete group that is finitely generated. This design relies on the rudimentary structure of the group's geometry. As a result, it has some anticipated characteristics, such as being unaffected by the selection of a word metric. We demonstrate that if R is a regular Noetherian ring with a limited global homological dimension and Γ has a finite asymptotic dimension and a finite model for the classifying space BΓ, then the Cartan map from the K-theory of R[Γ] to G-theory is a bijective mapping. However, our G-theory is more suitable for computing, as shown in a separate publication.The article presents findings and constructions that may be of independent interest, particularly in the context of constructing projective resolutions of finite type for certain modules over group rings.

Keywords – K-Theory, Group, G- Theory, Algebraic.

1. INTRODUCTION

Multiple functorial methods exist for associating an exact category with a given ring A. After completing that task, the K-theory of Quillen for the precise category provides a characteristic of the ring that remains unchanged.The most prevalent task falls under the classification of A-modules that are finitely produced and projective. The acceptable monomorphisms in this context refer to the inclusions of direct summands. Similarly, the admissible epimorphisms are the projections onto direct summands. The class of precise sequences refers to the collection of all short exact sequences that are divided in this particular circumstance. The outcome is referred to as the algebraic K-theory of A.Another valuable natural structure may be formed when the ring A is Noetherian. The category consists of all A-modules that are created by a limited number of elements. In this scenario, the acceptable monomorphisms refer to all injections that are linear with respect to A, whereas the acceptable epimorphisms refer to all surjections. The class of precise sequences is defined as the class that includes all short exact sequences. The outcome is referred to as the G-theory of A. Although these constructions and their applications are relevant, the prevailing perspective is that G-theory is a computable approximation of K-theory, focusing on the most often observed invariants. The relaxation of exactness criteria gives rise to natural Cartan maps from K-theory to G-theory. These maps facilitate the comparison of the two hypotheses. When the ring A is regular, it is well-established that the Cartan map is an isomorphism.

This study focuses on the situation of A = R[Γ], which is a group ring. The sole class of groups Γ for which it is known that the group rings are Noetherian, assuming The authors express gratitude for the financial assistance provided by the National Science Foundation.

The class of polycyclic-by-finite groups is characterized by the property of R being Noetherian. Philip Hall proposed a hypothesis stating that only groups that are polycyclic-by-finite have Noetherian integral group rings.

When A is not Noetherian, there is a trade-off. One may regard the collection of all A-modules that are created by a finite number of elements in the same way as before. There is a selection of precise sequences that are composed of small exact sequences, where all three modules are created in a limited manner. Indeed, this description suggests that the allowable monomorphisms consist only of injections, whereas the allowable epimorphisms consist only of surjections with a kernel that is finitely produced. A Cartan map exists from K-theory to the modified G-theory. Regrettably, the computational approaches are inadequate for this G-theory.

This study presents a novel alternative. The development will only focus on rings A that are group rings R[Γ], where R is a Noetherian ring and Γ is an arbitrary finitely generated group. This scenario is of particular relevance for applications in the field of geometric topology. We shall establish a precise structure on a complete subcategory B(R[Γ]) consisting of R[Γ]-modules that are created by a limited number of elements. It lies between the split structure and the modified form mentioned before. Due to its greater resemblance to K-theory rather than G-theory, we shall denote the new theory as I(R[Γ]). For those who subscribe to a particular narrative regarding the genesis of the conventional notation, it is worth mentioning that the letter "I" can also be found positioned between the two outermost letters in the name Grothendieck, and this placement is deliberately closer to the letter "K". To fully establish the analogy with G-theory, in a distinct paper [10], we demonstrate that the newly proposed I-theory can indeed be effectively computed as the homology of Γ with coefficients in the customary G-theory of R, under the assumption that Γ possesses a finite classifying space BΓ. This is achieved by demonstrating that a suitable assembly map is a kind of equivalence.In the latter part of this work, we calculate the Cartan map that connects the K-theory of R[Γ] to the new I-theory. Based on our prior research [7], we demonstrate that it is an equivalence when certain particular but broadly applicable algebraic and geometric assumptions are fulfilled.

Main Theorem. *Suppose* R *is a regular Noetherian ring of finite global homo- logical dimension. Suppose* Γ *is a discrete group with finite asymptotic dimension and a finite CW-model for* BΓ. *Then the Cartan map* $K_n(R[\Gamma]) \to I_n(R[\Gamma])$ *is an isomorphism in all dimensions* $n \in \mathbb{Z}$.

Our technique of using the coarse geometry of the group Γ to construct finite projective resolutions of individual modules from **B**(R[Γ]) should be of independent interest.

2. LITERATRE REVIEW

The I-theory may be seen as the spectrum of fixed points in the limited G-theory that we established in references [9, 11]. The basic definition requires a suitable metric space X and a Noetherian ring R as input. Following a short examination, we will focus on the specific scenario discussed in this work, where the group Γ is considered as a metric space based on word length, using a predetermined finite set of generators.

For a proper metric space X and a general ring R, Pedersen and Weibel define the category B(X, R) of *geometric* R-*modules over* X in [29]. The objects of this category are locally finite functions $F : X \to \mathbf{Free}_{fg}(R)$, from points of X to the category of finitely generated free R-modules. We will denote by F_x the module assigned to the point x and the object itself by writing down the collection $\{F_x\}$. The *local finiteness* condition requires that for every bounded subset $S \subset X$ the restriction of f to S has finitely many nonzero modules as values.

Let d be the distance function in X. The morphisms $\varphi\colon F \to F'$ in B(X, R) are collections of R-linear homomorphisms $\varphi_{x,x'} : F_x \longrightarrow F'_{x'}$, for all x and x′ in X, with the property that $\varphi_{x,x'}$ is the zero homomorphism whenever $d(x, x') > D$ for some fixed real number $D = D(\varphi) \geq 0$. In this case, we say that φ is *bounded by*

The sum in the formula is finite because of the local finiteness of $\{F'\}$.

The category B(X, R) is an additive category. The associated K-theory is called

the *bounded* K-*theory*. It is clear that a free action on X by isometries gives aninduced free action on the category B(X, R). What we need to do next is modify it

into an equivariant theory with useful fixed objects. For details we refer to chapterVI of [6].

Let ***E*Γ** be the category with the object set Γ and the unique morphism $\mu\colon \gamma_1 \to \gamma_2$ for any pair $\gamma_1, \gamma_2 \in \Gamma$. There is a left Γ-action on ***E*Γ** induced by the left multiplication in Γ. If C is a small category with a left Γ-action, then the category of functors $C_\Gamma = \mathrm{Fun}(\boldsymbol{E\Gamma}, C)$ is another category with the Γ-action given on objects by the formulas $\gamma(F)(\gamma') = \gamma F(\gamma^{-1}\gamma')$ and $\gamma(F)(\mu) = \gamma F(\gamma^{-1}\mu)$. It is always nonequivariantly equivalent to C. The fixed subcategory $\mathrm{Fun}(\boldsymbol{E\Gamma}, C)^\Gamma \subset C_\Gamma$ consists

of equivariant functors and equivariant natural transformations.

Explicitly, when C = B(X, R) with the Γ-action described above, the objects of $B_\Gamma(X, R)^\Gamma$ are the pairs (F, ψ) where $F \in B(X, R)$ and ψ is a function on Γ with $\psi(\gamma) \in \mathrm{Hom}(F, \gamma F)$ such that $\psi(1) = 1$ and $\psi(\gamma_1\gamma_2) = \gamma_1\psi(\gamma_2)\psi(\gamma_1)$. These conditions imply that ψ(γ) is always an isomorphism. The set of morphisms $(F, \psi) \to (F', \psi')$ consists of the morphisms $\varphi\colon F \to F'$ in B(X, R) such that the squares

A more refined theory is obtained by replacing $B_\Gamma(X, R)$ with the full subcategory $B_{\Gamma,0}(X, R)$ of functors sending all morphisms of ***E*Γ** to homomorphisms such that the maps and their inverses are bounded by 0. So $B_{\Gamma,0}(X, R)^\Gamma$ consists of (F, ψ) with ψ(γ) of filtration 0 for all $\gamma \in \Gamma$.

The fixed point category $B_{\Gamma,0}(X, R)^\Gamma$ is clearly additive. Now we have designed

a spectrum K(X, R) as the (nonconnective delooping of) the K-theory spectrumof $B_{\Gamma,0}(X, R)$ with the desired property

$$K(X, R)^\Gamma = K \quad B_{\Gamma,0}(X, R)^\Gamma \quad \simeq K(R[\Gamma])$$

whenever Γ is *geometrically finite* in the sense that there is a finite complex of homotopy type BΓ. We will assume that Γ is geometrically finite in the rest of thispaper.

Now let R be a Noetherian ring. At the basic level bounded G-theory is locally modeled on the category of finitely generated R-modules where the exact sequences are not necessarily split.

We will use the notation P(X) for the power set of X partially ordered by

inclusion. Let **Mod**(R) denote the category of left R-modules. If F is a left R- module, let **I**(F) denote the family of all R-submodules of F partially ordered by inclusion. An X-*filtered* R-module is a module F together with a functor $P(X) \to \mathbf{I}(F)$ such that the value on X is F. It is *reduced* if $F(\emptyset) = 0$.

We will use the notation where S[b] stands for the metric b-enlargement of S in

So, in particular, x[b] is the metric ball of radius b centered at x.

An R-homomorphism $f : F \to F'$ of X-filtered modules is *boundedly controlled* if there is a fixed number $b \geq 0$ such that the image f(F(S)) is a submodule of $F'(S[b])$ for all subsets S of X. It is called *boundedly bicontrolled* if, for some fixed $b \geq 0$, in addition to inclusions of submodules $f(F(S)) \subset F'(S[b])$, there are inclusions $f(F) \cap F'(S) \subset f F(S[b])$ for all subsets $S \subset X$.

There is a notion of an admissible exact sequence of filtered modules which is an

exact sequence of R-modules with the monic required to be a boundedly bicontrolled monomorphism and the epi required to be a boundedly bicontrolled epimorphism. For details we refer to section 3 of [11].

F is called *insular* or d-*insular* if there is a number $d \geq 0$ such that

$F(S) \cap F(U) \subset F(S[d] \cap U[d])$

for every pair of subsets S, U of X.

The category **LI**(X, R) of all lean insular reduced objects and boundedly con- trolled morphisms is exact with respect to the family of admissible exact sequences. This category has cokernels but not necessarily kernels.

An X-filtered R-module F is *locally finitely generated* if F(S) is a finitely gen- erated R-module for every bounded subset $S \subset X$. The category **B**(X, R) is thefull subcategory of **LI**(X, R) on the locally finitely generated objects. It is closed under extensions, so it is an exact category. There is an exact inclusion B(X, R) → **B**(X, R). The construction of $B_{\Gamma,0}(X, R)$ can be replicated here with the result $\mathbf{B}_{\Gamma,0}(X, R)$. Explicitly, the objects of $\mathbf{B}_{\Gamma,0}(X, R)$ are functors $\boldsymbol{E\Gamma} \to \mathbf{B}(X, R)$ such that all morphisms γ map to module isomorphisms $\psi(\gamma)$ in **B**(X, R) with theproperty that both $\psi(\gamma)$ and its inverse are bounded by 0.

3. COMPARISON TO K-THEORY

We will need to adapt and interpret some of the results from [7]. Those results are stated for equivariant Γ-modules F over a Noetherian ring R which are locally finite, lean, and satisfy a certain weakening of the insularity property. Here we restate and prove those results in the category **B**(R[Γ]).

Recall that the objects of **B**(R[Γ]), the *properly generated* R[Γ]-modules, are the finitely generated R[Γ]-modules which are lean and insular as filtered modules s(F, Σ) with respect a choice of the finite generating set Σ.

Definition 3.1. The group ring R[Γ] is *weakly regular Noetherian* if every properly generated R[Γ]-module has finite projective dimension.

Our main result in this paper confirms that the class of weakly regular Noetherian rings is surprisingly large and that, as a consequence, the spectrum I(R[Γ]) is weakly equivalent to the K-theory of R[Γ] for this class.

First we describe the geometric condition on the group Γ needed for this result.The asymptotic dimension is a coarse characteristic of a metric space introduced by M. Gromov [19].

Definition 3.2. One considers uniformly bounded families of subsets of a metricspace X. Such family C is called R-*disjoint* if for any pair of members U and U′ we have $U[R] \cap U' = \emptyset$. In these terms, X has *asymptotic dimension at most* n if for any $R > 0$ there exist $n + 1$ uniformly bounded R-disjoint families which together cover X. It is known that this property is a quasi-isometry invariant. In particular,when a finitely generated group is given a word metric, having *finite asymptotic dimension* (FAD) as a metric space is an invariant of the group itself.

The class of groups with FAD is very large. The following families of groups have been verified to have FAD: Gromov hyperbolic groups [19, 30], one-relator groups[2], virtually polycyclic groups [3], solvable groups with finite rational Hirsch length [16], Coxeter groups [14], cocompact lattices in connected Lie groups [7], arithmetic groups [22], S-arithmetic groups [23, 24], finitely generated linear group over a field of positive characteristic [20], relatively hyperbolic groups with the parabolic sub-group of finite asymptotic dimension [28], proper isometry groups of finite dimen- sional CAT(0) cube complexes, for example B(4)–T(4) small cancellation groups [33], mapping class groups [5], various generalized products of groups from these classes such as fundamental groups of developable complexes of asymptotically fi-nite dimensional groups [1].

We should mention here that there are known examples of asymptotically infi- nite dimensional groups, including Thompson's group F, Grigorchuk's group, and Gromov's group containing an expander. The reader can find a discussion of thesephenomena in [15].

Our proof of the main Theorem 3.8 is based on the following characterization of metric spaces of finite asymptotic dimension and a sequence of lemmas.

Definition 3.3. A map between metric spaces $\varphi: (M_1, d_1) \to (M_2, d_2)$ is a *uniform embedding* if there are two real functions f and g with $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$ such that $f(d_1(x, y)) \leq d_2(\varphi(x), \varphi(y)) \leq g(d_1(x, y))$ for all pairs of points x, y in M_1.

4. IMPLICATIONS AND SUGGESTIONS

One of the reasons we say that G(R[Γ]) seems to be close to the K-theory K(R[Γ]) is the usual in this area of algebra difficulty in constructing examples of properly generated

R[Γ]-modules that are not projective or even free. In this section, we want to show some examples in that direction. The following fact follows from the basic Theorem 2.18 in [9].

Proposition 4.1. *Suppose* R *is a Noetherian ring and* Γ *is a finitely generated group. The cokernels and images of boundedly bicontrolled homomorphisms between properly generated* R[Γ]*-modules are properly generated.*

A boundedly controlled idempotent of a filtered module is always bicontrolled. Indeed, if φ: F → F is an idempotent so that $\varphi^2 = \varphi$ then the restriction of φ to its image is the identity. Therefore φf(X) ∩ f(S) ⊂ φf(S). So, for example, the cokernels and images of idempotents of finitely generated free R[Γ]-modules are examples of properly generated R[Γ]-modules. The results are finitely generated projective R[Γ]-modules which are often filtered by non-projective R-submodules.

The following concrete non-projective examples exploit a similar idea.

Example 4.2. There are systematic constructions of non-free stably free Z[Γ]- modules for infinite discrete groups Γ, cf. section 17.4 of Johnson [25]. Specifi- cally, there are constructions for geometrically interesting groups by Dunwoody [17], Berridge–Dunwoody [4] for the trefoil group $\langle x, y \mid x^3 = y^2\rangle$ and similar groups in the work concerned with relation modules. The recent constructions of Harlander– Jensen [21] apply to the Baumslag–Solitar group $G = \langle x, y \mid xy^2x^{-1} = y^3\rangle$. It is

also true that G is generated by x and $x = y^4$. Let $K = R(G, x, x)$ be the kernel

of the canonical map $F_2 \to G$ sending the two free generators to x and x. The abelianization $M(G, x, x) = K/[K, K]$ is a Z[G]-module, where the action of G is by conjugation, called the relation module associated with the generating set $\{x, x\}$. Now Harlander and Jensen show that M is non free but $M \oplus Z[G] \cong Z[G]^2$. The pro-

jection onto M is an interesting idempotent e of $Z[G]^2$. Let $e_n : Z_n[G]^2 \to Z_n[G]^2$ be the reduction modulo $n > 1$. The image M_n of e_n is not flat over Z[G], so it is

not projective. However, e_n is a bicontrolled endomorphism of a properly generated

Z[G]-module $Z_n[G]^2$, so M_n is properly generated.

5. CONCLUSION

The study of modules over infinite group rings lies at the intersection of algebraic K-theory, geometric group theory, and homological algebra. In this work, we investigate structural properties of finitely generated modules over group rings $R[\Gamma]$, where R is a regular Noetherian ring of finite global dimension and Γ is an infinite, finitely generated discrete group. Using coarse geometric techniques and filtration methods arising from the word metric on Γ, we construct a well-behaved exact category of filtered $R[\Gamma]$–modules and compare its G-theory with the algebraic K-theory of R. Under suitable finiteness conditions on the asymptotic dimension of Γ, we obtain isomorphism and injectivity results for assembly-type maps, yielding rigidity and cancellation phenomena for projective and stably free modules over infinite group rings. These results contribute to a better understanding of when geometric properties of an infinite group are reflected in the module and K-theoretic invariants of its group ring.

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